TABLE 3.5 Comparison of Errors in Adams-Bashforth Methods

$y' = -y$, $y(0) = 1$, $t = 5$

h	Second order	Third order	Fourth order	Fifth order
2^{-5}	138850−10	−393711−12	115388−13	−344260−15
2^{-6}	344884−11	−487010−13	710535−15	−105524−16
2^{-7}	859472−12	−605583−14	440792−16	−326583−18
2^{-8}	214530−12	−754999−15	274471−17	−101563−19

$y' = -y^2$, $y(0) = 1$, $t = 5$

h	Second order	Third order	Fourth order	Fifth order
2^{-5}	568533−10	−365723−11	345730−12	−427887−13
2^{-6}	141718−10	−460309−12	222750−13	−143096−14
2^{-7}	353759−11	−577419−13	141392−14	−462985−16
2^{-8}	883718−12	−723062−14	890643−16	−147251−17

$y' = -t(y+y^2)$, $y(0) = 1$, $t = 5$

h	Second order	Third order	Fourth order	Fifth order
2^{-5}	100382−12	−946960−14	900407−15	−795755−16
2^{-6}	240987−13	−112385−14	524529−16	−228893−17
2^{-7}	591872−14	−136871−15	316472−17	−685903−19
2^{-8}	146752−14	−168876−16	194334−18	−208970−20

TABLE 3.6 Comparison of Errors in Nystrom Methods

$y' = -y$, $y(0) = 1$, $t = 5$

h	Second order	Third order	Fourth order	Fifth order
2^{-5}	370653−09	245941−09	188803−08	353071−06
2^{-6}	477957−10	177666−10	116657−09	245409−06
2^{-7}	619175−11	118679−11	121595−10	182088−05
2^{-8}	817258−12	844988−13	235654−11	−180580−05

$y' = -y^2$, $y(0) = 1$, $t = 5$

h	Second order	Third order	Fourth order	Fifth order
2^{-5}	519885−09	489842−09	327484−08	246166−06
2^{-6}	710399−10	376276−10	198742−09	291459−07
2^{-7}	978744−11	260210−11	884250−11	205047−08
2^{-8}	141185−11	170325−12	128293−12	−628150−09

$y' = -t(y+y^2)$, $y(0) = 1$, $t = 5$

h	Second order	Third order	Fourth order	Fifth order
2^{-5}	−788635−06	−603084+69	−122600+50	−562878+69
2^{-6}	−483239−07	−154901−05	−766248+49	−502570+69
2^{-7}	−299164−08	−960616−06	−301542+69	−392004+69
2^{-8}	−182068−09	233003−06	−328154−05	−409594+69

3.3 IMPLICIT MULTISTEP METHODS

In the preceding section we have expressed y_{n+1} in terms of previously calculated ordinates and slopes. A formula similar to (3.11) or (3.13), which involves the unknown slope y'_{n+1} on the right hand side, can be obtained if we replace $f(t, y)$ in (3.6) by a polynomial which interpolates $f(t, y)$ at t_{n+1}, $t_n, \cdots, t_{n-k+1}$ for an integer $k > 0$. Let us assume that $f(t, y)$ has $k+1$ continuous derivatives. The Newton backward difference formula which interpolates at these $k+1$ points in terms of $u = (t-t_n)/h$ is given by

$$P_k(t_n+hu) = f_{n+1}+(u-1)\nabla f_{n+1}+\frac{(u-1)u}{2!}\nabla^2 f_{n+1}$$

$$+\ldots+\frac{(u-1)u(u+1)\ldots(u+k-2)}{k!}\nabla^k f_{n+1}$$

$$+\frac{(u-1)u(u+1)\ldots(u+k-1)}{(k+1)!}h^{k+1}f^{(k+1)}(\xi)$$

$$=\sum_{m=0}^{k}(-1)^m\binom{1-u}{m}\nabla^m f_{n+1}+(-1)^{k+1}\binom{1-u}{k+1}h^{k+1}f^{(k+1)}(\xi) \tag{3.14}$$

Substituting (3.14) into (3.6), we get

$$y(t_{n+1}) = y(t_{n-j})+h\int_{-j}^{1}\Big[\sum_{m=0}^{k}(-1)^m\binom{1-u}{m}\nabla^m f_{n+1}$$

$$+(-1)^{k+1}\binom{1-u}{k+1}h^{k+1}f^{(k+1)}(\xi)\Big]du$$

or

$$y(t_{n+1}) = y(t_{n-j})+h\sum_{m=0}^{k}\delta_m^{(j)}\nabla^m f_{n+1}+T_{k+1}^{*(j)} \tag{3.15}$$

where

$$T_{k+1}^{*(j)} = h^{k+2}\int_{-j}^{1}(-1)^{k+1}\binom{1-u}{k+1}f^{(k+1)}(\xi)\,du$$

$$\delta_m^{(j)} = \int_{-j}^{1}(-1)^m\binom{1-u}{m}du \tag{3.16}$$

Neglecting $T_{k+1}^{*(j)}$ in (3.15), we get

$$y_{n+1} = y_{n-j}+h\sum_{m=0}^{k}\delta_m^{(j)}\nabla^m f_{n+1} \tag{3.17}$$

where

$$\delta_0^{(j)} = 1+j$$

$$\delta_1^{(j)} = -\frac{1}{2}(1+j)^2$$

$$\delta_2^{(j)} = -\frac{1}{12}(1+j)^2(1-2j)$$

$$\delta_3^{(j)} = -\frac{1}{24}(1+j)^2(1-j)^2$$

$$\delta_4^{(j)} = -\frac{1}{720}(1+j)^2(19-38j+27j^2-6j^3)$$

$$\delta_5^{(j)} = -\frac{1}{1440}(1+j)^2(27-54j+45j^2-16j^3+2j^4)$$

If we replace the difference operator $\nabla^m f_{n+1}$ in terms of the function values, we obtain

$$y_{n+1} = y_{n-j}+h\sum_{m=0}^{k}\delta_m^{*(j)}f_{n-m+1} \tag{3.18}$$

From (3.17) or (3.18) we can obtain a number of multistep formulas for various values of j. It is obvious from (3.15) that the implicit multistep methods are of one order higher than the corresponding explicit multistep methods with the same number of previously calculated ordinates and slopes.

3.3.1 Adams-Moulton formulas ($j = 0$)

Substituting $j = 0$ in (3.17), we get

$$y_{n+1} = y_n+h\left[f_{n+1}-\frac{1}{2}\nabla f_{n+1}-\frac{1}{12}\nabla^2 f_{n+1}-\frac{1}{24}\nabla^3 f_{n+1} -\frac{19}{720}\nabla^4 f_{n+1}-\frac{27}{1440}\nabla^5 f_{n+1}\cdots\right]$$

The error term associated with truncation after kth ∇ is

$$T_{k+1}^{*(0)} = h^{k+2}\int_0^1(-1)^{k+1}\frac{(u-1)u(u+1)\ldots(u+k-1)}{(k+1)!}f^{(k+1)}(\xi)\,du \tag{3.19}$$

Since the coefficient of $f^{(k+1)}(\xi)$ does not change sign in (0, 1), it is possible to write (3.19) as

$$T_{k+1}^{*(0)} = h^{k+2}\,\delta_{k+1}^{(0)}f^{(k+1)}(\xi)$$

The coefficients $\delta_m^{*(0)}$ in the formula

$$y_{n+1} = y_n+h\sum_{m=0}^{k}\delta_m^{*(0)}f_{n-m+1}$$

are given in Table 3.7.

3.3.2 Milne-Simpson formulas ($j = 1$)

These formulas can be obtained by substituting $j = 1$ in (3.17) and we find

$$y_{n+1} = y_{n-1}+h\left(2f_{n+1}-2\nabla f_{n+1}+\frac{1}{3}\nabla^2 f_{n+1}+0\nabla^3 f_{n+1} -\frac{1}{90}\nabla^4 f_{n+1}-\frac{1}{90}\nabla^5 f_{n+1}-\cdots\right) \tag{3.20}$$

The coefficients $\delta_m^{*(1)}$ of formula (3.18) are listed in Table 3.8.

3.5 GENERAL LINEAR MULTISTEP METHODS

Let us consider the general linear multistep methods of the form

$$y_{n+1} = a_1 y_n + a_2 y_{n-1} + \dots + a_k y_{n-k+1} + h(b_0 y'_{n+1} + b_1 y'_n + \dots + b_k y'_{n-k+1}) \quad (3.26)$$

or

$$y_{n+1} = \sum_{i=1}^{k} a_i y_{n-i+1} + h \sum_{i=0}^{k} b_i y'_{n-i+1}$$

Symbolically, we can write (3.26) as

$$\rho(E)\, y_{n-k+1} - h\sigma(E)\, y'_{n-k+1} = 0$$

where ρ and σ are polynomials defined by

$$\rho(\xi) = \xi^k - a_1\xi^{k-1} - a_2\xi^{k-2} - \dots - a_k$$

$$\sigma(\xi) = b_0\xi^k + b_1\xi^{k-1} + \dots + b_k$$

The above formula (3.26) can only be used if we know the values of the solution $y(t)$ and $y'(t)$ at k successive points. These k values will be assumed to be given. Further, if $b_0 = 0$, the resulting equation is called an explicit or predictor formula because y_{n+1} occurs only on the left hand side of the formula. In other words, y_{n+1} can be calculated directly from the right hand side values. If $b_0 \neq 0$, the equation is referred to as an implicit or corrector formula since y_{n+1} occurs in both sides of the equation. In other words the unknown y_{n+1} cannot be calculated directly since it is contained within y'_{n+1}. We can also assume that the polynomials $\rho(\xi)$ and $\sigma(\xi)$ have no common factors since, otherwise, (3.26) can be reduced to an equation of lower order. In order that the difference equation (3.26) should be useful for numerical integration, it is necessary that (3.26) be satisfied with good accuracy by the solution of the differential equation $y' = f(t, y)$, when h is small for an arbitrary function $f(t, y)$. This imposes restrictions on the coefficients a_i and b_i.

With the difference equation (3.26), we associate the difference operator L defined by

$$L[y(t), h] = y(t_{n+1}) - \sum_{i=1}^{k} a_i y(t_{n-i+1}) - h\sum_{i=0}^{k} b_i y'(t_{n-i+1}) \quad (3.27)$$

We assume that the function $y(t)$ has continuous derivatives of sufficiently high order. Expanding $y(t_{n-i+1})$ and $y'(t_{n-i+1})$ in Taylor's series, we have

$$y(t_{n-i+1}) = y(t_n) + (1-i)hy'(t_n) + \frac{(1-i)^2}{2!} h^2 y''(t_n) + \dots + \frac{(1-i)^p}{p!} h^p y^{(p)}(t_n) + \frac{1}{p!} \int_{t_n}^{t_{n-i+1}} (t_{n-i+1} - s)^p y^{(p+1)}(s)\, ds$$

$$y'(t_{n-i+1}) = y'(t_n)+(1-i)hy''(t_n)+\frac{(1-i)^2}{2!}h^2y'''(t_n)$$

$$+\ldots+\frac{(1-i)^{p-1}}{(p-1)!}h^{p-1}y^{(p)}(t_n)$$

$$+\frac{1}{(p-1)!}\int_{t_n}^{t_{n-i+1}}(t_{n-i+1}-s)^{p-1}y^{(p+1)}(s)\,ds$$

Substituting in (3.27), we get

$$L[y(t), h] = C_0y(t_n)+C_1hy'(t_n)+C_2h^2y''(t_n)+\cdots +C_ph^py^{(p)}(t_n)+T_n \tag{3.28}$$

where

$$C_0 = 1-\sum_{i=1}^{k} a_i$$

$$C_q = \frac{1}{q!}[1-\sum_{i=1}^{k} a_i(1-i)^q]-\frac{1}{(q-1)!}\sum_{i=0}^{k} b_i(1-i)^{q-1},\quad q = 1, 2,\ldots, p$$

$$T_n = \frac{1}{p!}\Bigg[\int_{t_n}^{t_{n+1}}(t_{n+1}-s)^py^{(p+1)}(s)\,ds$$

$$-\sum_{i=1}^{k} a_i\int_{t_n}^{t_{n-i+1}}(t_{n-i+1}-s)^py^{(p+1)}(s)\,ds$$

$$-hp\int_{t_n}^{t_{n+1}} b_0(t_{n+1}-s)^{p-1}y^{(p+1)}(s)\,ds$$

$$-hp\sum_{i=1}^{k} b_i\int_{t_n}^{t_{n-i+1}}(t_{n-i+1}-s)^{p-1}y^{(p+1)}_{(s)}ds\Bigg] \tag{3.29}$$

DEFINITION 3.1 The difference operator (3.27) and the associated linear multistep method (3.26) are said to be of order p if, in (3.28)

$$C_0 = C_1 = C_2 = \ldots = C_p = 0 \text{ and } C_{p+1} \neq 0 \tag{3.30}$$

Thus for any function $y(t) \in C^{(p+2)}$ and for some nonzero constant C_{p+1}, we have

$$L[y(t), h] = -C_{p+1}h^{p+1}y^{(p+1)}_{(t_n)} + 0(h^{p+2}) \tag{3.31}$$

where $C_{p+1}/\sigma(1)$ is called the *error constant*.

In particular, $L[y(t), h]$ vanishes identically when $y(t)$ is a polynomial whose degree is less than or equal to p. We now introduce the following definitions.

DEFINITION 3.2 The linear multistep method (3.26) is said to be consistent if it has order $p \geqslant 1$.

DEFINITION 3.3 The linear multistep method (3.26) is said to satisfy the *root condition* if the roots of the equation $\rho(\xi) = 0$ be inside the unit circle in the complex plane, and are simple if they lie on the circle.

We shall now use the definitions of order, consistency, and root condition to determine the parameters a_i and b_i in the linear multistep method (3.26).

3.5.1 Determination of a_i and b_i

Equation (3.31) holds good for any function $y(t) \in C^{(p+2)}$. The constants C_i and p are independent of $y(t)$. These can thus be determined by a particular case $y(t) = e^t$, and substituting it in (3.31) we obtain

$$\begin{aligned} L[(e^t, h] &= e^{t_{n+1}} - a_1 e^{t_n} - \ldots - a_k e^{t_{n-k+1}} \\ &\quad - h(b_0 e^{t_{n+1}} + b_1 e^{t_n} + \ldots + b_k e^{t_{n-k+1}}) \\ &= -C_{p+1} h^{p+1} e^{t_n} + 0(h^{p+2}) \end{aligned}$$

Simplifying we get

$$\begin{aligned} L[e^t, h] &= [(e^{kh} - a_1 e^{(k-1)h} - \ldots - a_k) - h(b_0 e^{kh} + b_1 e^{(k-1)h} + \ldots + b_k)] e^{t_{n-k+1}} \\ &= -C_{p+1} h^{p+1} e^{t_n} + 0(h^{p+2}) \end{aligned}$$

or

$$\rho(e^h) - h\sigma(e^h) \approx -C_{p+1} h^{p+1} + 0(h^{p+2})$$

Putting $e^h = \xi$, as $h \to 0$, $\xi \to 1$, the above equation becomes

$$\rho(\xi) - (\log \xi)\sigma(\xi) = -C_{p+1}(\xi - 1)^{p+1} + 0((\xi - 1)^{p+2}) \qquad (3.32)$$

or

$$\frac{\rho(\xi)}{\log \xi} - \sigma(\xi) = -C_{p+1}(\xi - 1)^p + 0((\xi - 1)^{p+1}) \qquad (3.33)$$

Equations (3.32) and (3.33) provide us with the methods for determining $\rho(\xi)$ or $\sigma(\xi)$ for maximum order if $\sigma(\xi)$ or $\rho(\xi)$ is given.

If $\sigma(\xi)$ is specified, (3.32) can be used to determine a $\rho(\xi)$ of degree k such that the order is at least k. The $(\log \xi)\ \sigma(\xi)$ can be expanded as a power series in $(\xi - 1)$ and the terms up to $(\xi - 1)^k$ can be used to find $\rho(\xi)$. If, on the other hand, we are given $\rho(\xi)$ we can use (3.33) to determine $\sigma(\xi)$ of degree $\leqslant k$ such that the order is at least $k+1$. The $\rho(\xi)/\log \xi$ is expanded as a power series in $(\xi - 1)$, and terms up to $(\xi - 1)^k$ are used to get $\sigma(\xi)$. For example, a few choices of the polynomial $\rho(\xi)$ and the resulting polynomials $\sigma(\xi)$ which give the well-known methods are as follows:

Adams-Bashforth Methods

$$\rho(\xi) = \xi^{k-1}(\xi - 1) \text{ and } \sigma(\xi) \text{ of degree } k-1$$

$$\sigma(\xi) = \xi^{k-1} \sum_{m=0}^{k-1} \gamma_m (1 - \xi^{-1})^m$$

where $$\gamma_m+\frac{1}{2}\gamma_{m-1}+\ldots+\frac{1}{m+1}\gamma_0 = 1,\ m = 0, 1, 2, \ldots$$

Nystrom Methods

$$\rho(\xi) = \xi^{k-2}(\xi^2-1) \text{ and } \sigma(\xi) \text{ of degree } k-1$$

$$\sigma(\xi) = \xi^{k-1}\sum_{m=0}^{k-1}\gamma_m(1-\xi^{-1})^m$$

where $$\gamma_m+\frac{1}{2}\gamma_{m-1}+\ldots+\frac{1}{m+1}\gamma_0 = \begin{cases} 2, & m = 0 \\ 1, & m = 1, 2, \ldots \end{cases}$$

Adams-Moulton Methods

$$\rho(\xi) = \xi^{k-1}(\xi-1) \text{ and } \sigma(\xi) \text{ of degree } k$$

$$\sigma(\xi) = \xi^{k}\sum_{m=0}^{k}\gamma_m(1-\xi^{-1})^m$$

where $$\gamma_m+\frac{1}{2}\gamma_{m-1}+\ldots+\frac{1}{m+1}\gamma_0 = \begin{cases} 1, & m = 0 \\ 0, & m = 1, 2, \ldots \end{cases}$$

Milne-Simpson Methods

$$\rho(\xi) = \xi^{k-2}(\xi^2-1) \text{ and } \sigma(\xi) \text{ of degree } k$$

$$\sigma(\xi) = \xi^{k}\sum_{m=0}^{k}\gamma_m(1-\xi^{-1})^m$$

where $$\gamma_m+\frac{1}{2}\gamma_{m-1}+\ldots+\frac{1}{m+1}\gamma_0 = \begin{cases} 2, & m = 0 \\ -1, & m = 1 \\ 0, & m = 2, 3, \ldots \end{cases}$$

As the number of coefficients in (3.26) is equal to $2k+1$ we may expect that they can be chosen so that $2k+1$ relations of the type (3.30) are satisfied, in which p is equal to $2k$. However, the root condition to be satisfied by the method considerably restricts this order.

We now state the fundamental theorem which specifies the maximum order of a linear k-step method.

THEOREM 3.1 *For any positive integer k although there exists a consistent method of order $p = 2k$, the order of a k-step method satisfying the root condition cannot exceed $k+2$. If k is odd it cannot exceed $k+1$.*

Example 3.3 Let $\rho(\xi) = (\xi-1)(\xi-\lambda)$ where λ is real and $-1 \leqslant \lambda < 1$, find $\sigma(\xi)$.

We have

$$\frac{\rho(\xi)}{\log \xi} = \frac{(\xi-1)[(1-\lambda)+(\xi-1)]}{\log(1+(\xi-1))}$$

$$\sigma(\xi) = 1-\lambda+\frac{3-\lambda}{2}(\xi-1)+\frac{5+\lambda}{2}(\xi-1)^2$$
$$-\frac{1+\lambda}{24}(\xi-1)^3+0((\xi-1)^4)$$

Note that for $\lambda \neq -1$, the order is 3 and for $\lambda = -1$, the order is 4.

3.5.2 Estimate of truncation error

We can write (3.29) as

$$T_n = \frac{1}{p!}\int_{t_{n-k+1}}^{t_{n+1}} \{\overline{(t_{n+1}-s)}^p - ph\, b_0\overline{(t_{n+1}-s)}^{p-1}$$
$$+\sum_{i=1}^{k} a_i\overline{(t_{n-i+1}-s)}^p + ph\, b_i\overline{(t_{n-i+1}-s)}^{p-1}\}\, y^{(p+1)}(s)\, ds$$
$$= \frac{1}{p!}\int_{t_{n-k+1}}^{t_{n+1}} G(s) y^{(p+1)}(s)\, ds$$

where $$\overline{(t_{n-i+1}-s)} = \begin{cases} t_{n-i+1}-s & \begin{cases} t_{n-i+1} \leqslant s \leqslant t_n & i \neq 0 \\ t_n \leqslant s & i = 0 \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

Substituting $u = (s-t_n)/h$, we get

$$T_n = \frac{h^{p+1}}{p!}\int_{-k+1}^{1} \{\overline{(1-u)}^p - p\, b_0\overline{(1-u)}^{p-1}$$
$$+\sum_{i=1}^{k} [a_i\overline{(1-i-u)}^p + p\, b_i\overline{(1-i-u)}^{p-1}]\}\, y^{(p+1)}(t_n+hu)\, du$$
$$= \frac{h^{p+1}}{p!}\int_{-k+1}^{1} G(u)\, y^{(p+1)}(t_n+hu)\, du \tag{3.34}$$

The function $G(u)$ is called the *influence function*. If $G(u)$ does not change sign over the interval of integration $[-k+1, 1]$, then we may write (3.34) in the form

$$T_n = \frac{h^{p+1}}{p!}\, y^{(p+1)}(\eta)\int_{-k+1}^{1} G(u)\, du \tag{3.35}$$

where $-k+1 < \eta < 1$. But if the influence function $G(u)$ does change sign over $[-k+1, 1]$, then the error cannot be expressed in the form (3.35), although we may bound the error as

$$| T_n | \leqslant \frac{h^{p+1}}{p!} | y^{(p+1)}(\eta) | \int_{-k+1}^{1} | G(u) | du$$

Example 3.4 Obtain a fifth order formula of the form

$$y_{n+1} = a_1 y_n + a_2 y_{n-1} + a_3 y_{n-2} + a_4 y_{n-3} + h(b_0 y'_{n+1} + b_1 y'_n + b_2 y'_{n-1})$$

Express each coefficient in terms of a_2. Calculate the explicit form of the error term.

Expanding each term about t_n in the Taylor series and equating the cofficients of h^0 through h^5 to zero, we get

$$1 = a_1 + a_2 + a_3 + a_4$$

$$1 = -a_2 - 2a_3 - 3a_4 + b_0 + b_1 + b_2$$

$$\frac{1}{2} = \frac{1}{2}(a_2 + 4a_3 + 9a_4) + b_0 - b_2$$

$$\frac{1}{6} = \frac{1}{6}(-a_2 - 8a_3 - 27a_4) + \frac{1}{2}(b_0 + b_2)$$

$$\frac{1}{24} = \frac{1}{24}(a_2 + 16a_3 + 81a_4) + \frac{1}{6}(b_0 - b_2)$$

$$\frac{1}{120} = \frac{1}{120}(-a_2 - 32a_3 - 243a_3) + \frac{1}{24}(b_0 + b_2)$$

The truncation error is given by

$$T_n = -\frac{h^6}{5!}\int_{-3}^{1} G(u)\, y^{(6)}(t_n + hu)\, du$$

where
$$G(u) = \begin{cases} (u-1)^5 + 5b_0(u-1)^4, & 0 \leqslant u \leqslant 1 \\ a_2(u+1)^5 - 5b_2(u+1)^4 + a_3(u+2)^5 + a_4(u+3)^5, & -1 \leqslant u \leqslant 0 \\ a_2(u+2)^5 + a_4(u+3)^5, & -2 \leqslant u \leqslant -1 \\ a_4(u+3)^5, & -3 \leqslant u \leqslant -2 \end{cases}$$

The coefficients a_i and b_i may be obtained as

$$306a_1 = -413a_2 + 468, \quad 34a_3 = 13a_2 - 20$$

$$153a_4 = -5a_2 + 9, \quad 34b_0 = -a_2 + 12$$

$$51b_1 = 31a_2 + 36, \quad 34b_2 = 37a_2 - 36$$

This is a linear difference equation of order two with constant coefficients. Its characteristic polynomial is

$$\xi^2-4\xi+(3+2\bar{h}) = 0$$

which has roots

$$\xi_{1h} = 2-\sqrt{(1-2\bar{h})}$$

and
$$\xi_{2h} = 2+\sqrt{(1-2\bar{h})}$$

The general solution of (3.37) may be written as

$$y_n = C_1\,\xi_{1h}^n+C_2\,\xi_{2h}^n \tag{3.38}$$

where C_1 and C_2 are two arbitrary constants. Choosing the conditions

$$y_0 = 1, \quad y_1 = z_1$$

where z_1 is still unspecified, we obtain

$$C_1 = (\xi_{2h}-z_1)/(\xi_{2h}-\xi_{1h}),\ C_2 = (z_1-\xi_{1h})/(\xi_{2h}-\xi_{1h})$$

We now study the asymptotic behaviour of y_n as $h \to 0$ and $n \to \infty$ while $nh = t$ remains fixed. Let us choose $y_1 = e^{\bar{h}}$, which is the value of the exact solution at $t = h$, satisfying $\lim_{h\to 0} y_1 = 1$. Now we get

$$\begin{aligned}
\xi_{1h} &= 1+\bar{h}+\frac{1}{2}\bar{h}^2+0(\bar{h}^3)\\
\xi_{2h} &= 3-\bar{h}+0(\bar{h}^2)\\
\xi_{2h}-\xi_{1h} &= 2\sqrt{(2-\bar{h})}\\
z_1-\xi_{1h} &= e^{\bar{h}}-\xi_{1h} = -\frac{1}{3}\bar{h}^3+0(\bar{h}^4)\\
\xi_{2h}-z_1 &= \xi_{2h}-e^{\bar{h}} = 2+0(\bar{h})\\
\xi_{1h}^n &= e^{\lambda t}+0(\bar{h})\\
\xi_{2h}^n &= 3^n\left(\exp\left(-\frac{1}{3}\lambda t\right)+0(\bar{h})\right)
\end{aligned}$$

where $nh = t$, fixed.

Therefore, the solution (3.38) may be written as

$$y_n = \frac{1}{2\sqrt{1-2\bar{h}}}(\exp(\lambda t)+0(\bar{h}))(2+0(\bar{h}))$$
$$+\frac{1}{2\sqrt{1-2\bar{h}}}3^n\left(\exp\left(-\frac{1}{3}\lambda t\right)+0(\bar{h})\right)\left(-\frac{1}{3}\bar{h}^3+0(\bar{h}^4)\right)$$

As $h \to 0$, we see that the first term converges to the exact solution $\exp(\lambda t)$. The second term behaves asymptotically like

$$-\frac{1}{6}\lambda^3 t^3\exp\left(-\frac{1}{3}\lambda t\right)(3^n/n^3) \text{ as } n\to\infty$$

Hence $y_n \to -\infty$. The method (3.36) is not convergent for the initial value problem $y' = \lambda y$, $y(0) = 1$. Next, we assume $y_1 = e^{\bar{h}}+h^2$ and obtain a numerical solution $\bar{y}_n$ of (3.37) which behaves asymptotically like $\frac{1}{2}t^2$

$\exp(-\lambda t)(3^n/n^2)$ as $n \to \infty$, $nh = t$ fixed. We find $y_n - \bar{y}_n \to -\infty$ as $h \to 0$ while the perturbation h^2 in y_n also approaches zero. Therefore, Definition 3.4 cannot be satisfied for any h_0.

3.5.4 Other stability results

In order to discuss the stability in a quantitative way, we must define a differential equation as well as the numerical method used in the approximate solution. The equation studied in this connection is the simple linear first order differential equation

$$y' = \lambda y, \quad y(t_0) = y_0$$

where λ may be a complex number. The exact solution for this equation at $t = t_n$ is given by

$$y(t_n) = y(t_0)\, e^{\lambda nh} = y(t_0)\,(e^{\lambda h})^n \tag{3.39}$$

Ignoring the round-off errors, the computed solution satisfies

$$y_{n+1} = \sum_{i=1}^{k} a_i\, y_{n-i+1} + h\lambda \sum_{i=0}^{k} b_i\, y_{n-i+1} \tag{3.40}$$

The true solution satisfies

$$y(t_{n+1}) = \sum_{i=1}^{k} a_i\, y(t_{n-i+1}) + h\lambda \sum_{i=0}^{k} b_i\, y(t_{n-i+1}) + T_n \tag{3.41}$$

where T_n is the local truncation error.

Subtracting (3.41) from (3.40) and substituting $\epsilon_n = y_n - y(t_n)$, we get

$$\epsilon_{n+1} = \sum_{i=1}^{k} a_i\, \epsilon_{n-i+1} + h\lambda \sum_{i=0}^{k} b_i\, \epsilon_{n-i+1} - T_n \tag{3.42}$$

or

$$(\rho(E) - h\lambda\sigma(E))\, \epsilon_{n-k+1} + T_n = 0 \tag{3.43}$$

This is a kth order, linear, nonhomogeneous, difference equation with constant coefficients. If the estimates of $\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}$ are available and T_n is known, the difference equation can be solved for all n. Let as assume that T_n is constant and is equal to T. The solution of (3.43) will consist of a particular solution plus a linear combination of the independent solution of the homogeneous equation with $T = 0$. The homogeneous equation is

$$(\rho(E) - h\lambda\sigma(E))\, \epsilon_{n-k+1} = 0 \tag{3.44}$$

We seek the solution of (3.44) in the form

$$\epsilon_n = A\xi^n \tag{3.45}$$

where ξ is to be determined and A is a constant. Substituting (3.45) in (3.44), we get

$$A(\rho(\xi) - h\lambda\sigma(\xi))\, \xi^{n-k+1} = 0$$

or

$$\rho(\xi) - h\lambda\sigma(\xi) = 0 \tag{3.46}$$

The general solution of the difference equation (3.43) for distinct roots can be written as

$$\epsilon_n = A_1\xi_{1h}^n + A_2\xi_{2h}^n + \ldots + A_k\xi_{kh}^n + \frac{T}{h\lambda\,\rho'(1)} \tag{3.47}$$

The interval of absolute stability is listed in Table 3.10.

TABLE 3.10 INTERVAL OF ABSOLUTE STABILITY ON REAL LINE

k	1	2	3	4	5
		Adams-Bashforth methods			
$(\beta, 0)$	−2	−1.33	−0.55	−0.3	−0.2
		Adams-Moluton methods			
$(\beta, 0)$	$-\infty$	$-\infty$	−6	−3.0	−1.8

The linear multistep methods having the interval of absolute stability $(-\infty, 0)$ are called *A-stable* methods. Here, we have $|\xi_{jh}| < 1, j = 1(1)k$.

DEFINITION 3.8 A linear multistep method when applied to the differential equation of the form $y' = \lambda y$ and λ is a (complex) constant with negative real part is called *A-stable* if all solutions of (3.26) tend to zero, as $n \to \infty$.

The region of stability is shown in Figure 3.2.

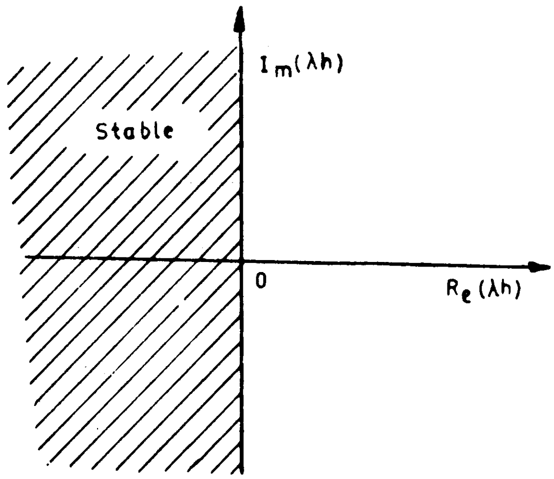

Fig. 3.2 Stability region

We now state the main result about *A*-stable linear multistep methods.

THEOREM 3.6 *The order p, of an A-stable linear multistep method cannot exceed 2 and the method must be implicit.*

If we use the trapezoidal formula or the second order Adams-Moulton method

$$y_{n+1} = y_n + \frac{h}{2}(y'_{n+1} + y'_n) \tag{3.52}$$

to approximate $y' = \lambda y$, then Equation (3.52) becomes

$$(2-\bar{h})\, y_{n+1} - (2+\bar{h})\, y_n = 0 \tag{3.53}$$

The characteristic Equation of (3.53) is given by

$$(2-\bar{h})\, \xi - (2+\bar{h}) = 0$$

The solution of the difference Equation (3.53) can be written as

$$y_n = \bar{c}_1 \left(\frac{2+\bar{h}}{2-\bar{h}} \right)^n$$

The root ξ_{1h} is shown in Figure 3.3.

From Figure 3.3, it is obvious that the trapezoidal formula is stable for all values of $\bar{h}$. Similarly, we can show that the backward Euler method

$$y_{n+1} = y_n + h\, y'_{n+1}$$

is also stable for all values of $\bar{h}$.

Thus, the trapezoidal and backward Euler methods are *A*-stable.

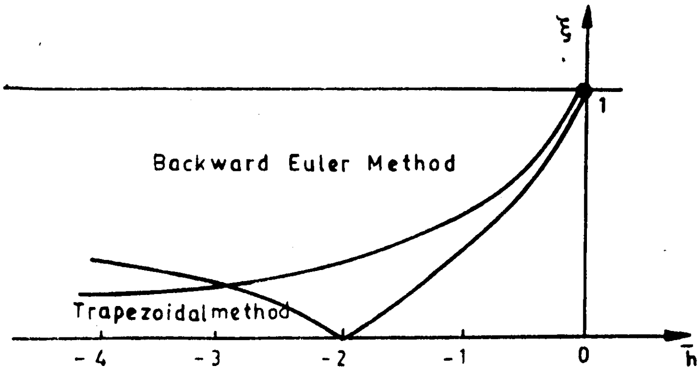

Fig. 3.3 Roots of trapezoidal and backward Euler methods

The *A*-stable linear multistep methods are very useful for integrating stiff systems of ordinary differential equations. Unfortunately, the class of *A*-stable linear multistep methods is rather small. A natural weakening of the stability requirement is to demand that the absolute stability condition

The value $r = 5$ does not give eleventh order method. There are no twelfth order and higher order stiffly stable multistep methods of this type.

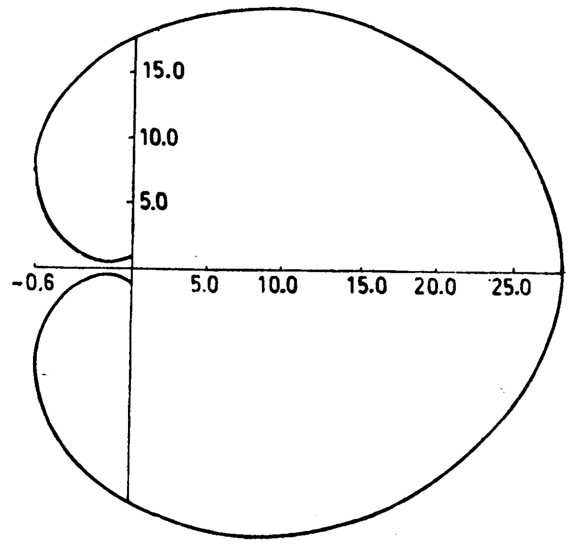

Fig. 3.5 Locus of $\rho(\xi)/\sigma(\xi)$, $\xi = \exp(i\theta)$, $\theta \in [0, 2\pi]$

DEFINITION 3.10 The linear multistep method (3.26) is said to be relatively stable if

$$| \xi_{jh} | \leqslant | \xi_{1h} |, j = 2, 3, \ldots, k$$

The region of relative stability is defined to be a set of points in the λh-plane for which the method is relatively stable.

It may be pointed out here that absolute stability does not mean relative stability, because we may have

$$| \xi_{jh} | \leqslant 1, j = 1, 2, \ldots, k \text{ but } | \xi_{1h} | \leqslant | \xi_{jh} |, j = 2, 3, \ldots, k$$

To illustrate the difference between absolute and relative stabilities, let us apply the third order Adams-Moulton method

$$y_{n+1} = y_n + \frac{h}{12}(5y'_{n+1} + 8y'_n - y'_{n-1}) \tag{3.54}$$

to the initial value problem $y' = \lambda y$, $y(t_0) = y_0$. Equation (3.54) becomes

$$\left(1 - \frac{5}{12}\bar{h}\right) y_{n+1} - \left(1 + \frac{2}{3}\bar{h}\right) y_n + \frac{\bar{h}}{12} y_{n-1} = 0 \tag{3.55}$$

where $\bar{h} = \lambda h$.

Equation (3.55) is a second order difference equation which will give one extraneous solution. We are concerned with the rate of growth of this